ON CONFORMALLY-INVARIANT MOTIONS OF A MATERIAL POINT

(O KONFORMNO-INVARIANTNYKH DVIZHENIIAKH MATERIAL'NOI TOCHKI)

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One of the invariance principles used in theoretical physics is applied to the mass-independent motions of a material point: A law of material-point motion is constructed which admits the broadest group of space-time transformations relative to which the Maxwell equations are invariant, as the maximum group.

The kinematics of conformally-invariant motions of a material point is described. In connection with one of the possible treatments of such motions, a comparison is made with the Galileo inertia law.

1. Formulation of the problem. Let us consider Maxwell's equations for a matter-free space

$$\operatorname{rot} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{div} \mathbf{E} = 0, \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div} \mathbf{H} = 0 \qquad (1.1)$$

where c is a universal constant (the velocity of light).

It is known that these equations are invariant relative to a conformal fifteen parameter group of space-time transformations.

Henceforth, we shall name this group the group G_{15}.

It is necessary to construct a family of motions which is transformed into itself for all transformations of the group G_{15} and just this group, and also to indicate its possible interpretation.

Invariance principles. Geometric and dynamical invariance principles play a significant part in modern theoretical physics. Following Wigner [1 and 2], let us briefly elucidate their substance by the example of simpler (and older) geometrical principles.

Let some law of nature connecting some physical quantities to the space coordinates and time be established. Evidently, if the form of this law were to be altered from point to point, say, no observer would be able to observe it. Therefore, the very existence of this law already contains the fact of its invariance relative to some coordinate and time transformations (relative to coordinate transformations in this case). For this reason, we may be justified in assuming that every other law is also invariant relative to the same coordinate transformations. The set of all such transformations forms a group. Therefore, all laws, or at least some sets of them should be associated with some general group of transformations. The crux of the invariance principles is indeed this constraint imposed on the laws: just as the laws of nature check the regularities of the events they describe, so do the invariance principles establish a check on the laws themselves. The connection between the laws of nature and the events on one hand, and between the invariance principles and the laws of nature on the other, also contains many other similarities.

The great heuristic value of the invariance principles is brought to light by: firstly, the fact that the truth of each new law of nature is confirmed by its covariance relative to an already known group of transformations (the Lorentz group, say); and secondly, the fact that some properties of such laws of nature which have not been written down explicitly may be established by using invariance principles. The idea of the invariance of the laws of nature dates actually back to Galileo. Einstein however was first to grasp its full implications and to shed new light on Galileo's relativity principle [3]. A description of gravitational interaction is given in the Einstein theory of gravitation (general theory of relativity [3]). Weyl attempted to reduce all kinds of interactions to gravitational and electromagnetic ones connecting the latter with the space metric (see [4], for example). In fact both theories are closely connected with the idea of dynamical invariance. At present such concepts are successfully applied to quantum-mechanical phenomena, in theories of elementary particle interactions (see [5], say), and they lead to very interesting and promising physical consequences. Theoretical aspects of the problem are also developed. For example, it is indicated how, from a knowledge of the group, to construct some further relationships [6].

A deeper and more thorough exposition of the above ideas may be found in the aforementioned work of E. Wigner [1 and 2]. An extensive bibliography may also be found there.

*Maxwell equations and invariance**. The system of Maxwell equations (0.1) is closed in the sense that it may not be supplemented essentially by any other relationship between the electric and magnetic field intensities, not derived from these equations themselves. Furthermore, it seems evident that any attempt to clarify the 'fine structure' of these equations, to refine them, will inevitably lead to the introduction of new physical quantities in the equations, and perhaps, new universal constants as well. It hence follows that within the scope of the $(t, x, \mathbf{E}, \mathbf{H}, c)$ concepts, and only within these concepts, the Maxwell equations are absolutely exact and remain so throughout all time. Therefore, the Maxwll equations describe some closed universe in which only space-time and the electromagnetic field are defined.

However, the most natural and profound description of the properties of this universe is given not by the Maxell equations themselves, but by the maximum Lie group of transformations which conserves these equations. In fact, to characterize a physical quantity

* In the author's opinion the viewpoint developed here agrees with Klein's Erlangen program.

means, apparently, to show how this quantity changes its state upon interaction with other physical quantities. The latter is achieved by comparing possible states between themselves. This identification process is realized mathematically by a group of transformations. In particular, a group of transformations of the space-time coordinates will define the space-time configuration, and will help to understand it. Formally, the space coordinates and time are completely equivalent to other physical quantities in the Maxwell equations. Actually, their chief advantage in their application is, that they form a basis for quantitative measurement. As V.A. Fock [7] (p. 207) remarks, even if the physical meaning of the quantities t and x were not known in advance, the equations in which they appear would provide an insight into their properties. If the Maxwell equations are valid, we therefore know, at least in principle, about the physical properties and interactions of the set of variables $(t, x, \mathrm{E}, \mathrm{H}, c)$.

In this case, we must accept the fact that every other law expressing the relationship between elements of the same set of quantities, or part of them, does not provide us with any new information about them. The group associated with this law, which uniquely characterizes its universe, may not be of greater or lesser order, than the group of Maxwell equations. It may not be greater since this would imply the presence of properties of the $(t, x, \mathrm{E}, \mathrm{H}, c)$ universe not described by the Maxwell equations.

It may not also be of lesser order since in this case the group would allow new invariants differing from the system of Maxwell equations, and the latter would not then be closed. Therefore, the maximum group of this law should coincide exactly with the group of Maxwell equations.

Maxwell equations and the law of inertia. The Galileo inertia law belongs to the category of laws connecting part of the variables of the set t, x, E, H, namely t and x. It would certainly be absurd to hope that an experimental law resulting from observations having very little in common with those which led to the discovery of the fundamental properties of electromagnetism, should be associated with the same group as that of the Maxwell equations (in fact, the maximum group transforming the family of uniform and rectilinear motions into itself, is a projective group). However, since Maxwell's equations include the quantities characterising the electromagnetic field, hence Maxwellian universe turns out to be broader than the Galilean one. Hence, it may be expected that the Maxwell equations already contain the Galileo law, and moreover, that they may contribute additional information about the motion of a free material point, not deducible from Galileo's law. That it is indeed so, is shown below (section 4).

Such is the heuristic reasoning leading to the setting up of the problem formulated at the beginning of this section, and to the interpretation of conformally-invariant motions as motions of an isolated point.

The conformal group G_{15}. This group is well known. Liouville [8] showed that any conformal mapping of a space of more than two dimensions reduces to a sequence of inversions and similarity transformations. Lie [9] later gave another proof of this theorem. In 1909 Cunningham and Bateman [10] showed that the Maxwell equations are invariant relative to conformal mappings. Apparently Klein [11] (p. 95) was the first to notice the possibility of utilising the group G_{15} in the manner similar to that of the Lorentz group. In 1936, Page [12], guided by the ideas of Milne [13], constructed a 'new relativity' admitting of the possibility of uniformly accelerated motion and dispensing with the concepts of rigid scales and clocks. Still earlier, Caratheodory, in order to unify the axiomatics of the theory of relativity, constructed a theory utilizing only the properties of light signals. In this work [14] he showed that non-linear transformations should be excluded from the group G_{15}, by using a first approximation to show that one of these transformations brings an isolated point at rest into the state of uniformly variable rectilinear motion, in contradiction with known mechanical facts. Conformally-invariant analogs of the equations of quantum mechanics, the Dirac equations, and the Lorentz equations for an electron are constructed in [15].

In one of the important Wigner works [2], already cited, invariance relative to the group G_{15} is listed among the other invariance principles playing an important part in physics.

2. Finding the group G_{15}. We shall use a condition, derived from Maxwell's equations, of invariance of the system of equations

$$ds^2 \equiv (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2\, dt^2 = 0, \quad \frac{d^2x^i}{dt^2} = 0 \qquad (i = 1, 2, 3) \quad (2.1)$$

describing the law of rectilinear and uniform propagation of light beams*. That the velocity of light c is constant, is expressed mathematically by the fact that the universal constant c will remain constant under the transformations.

Let us find the Lie algebra of infinitesimal operators corresponding to the group G_{15}. Let τ be a canonical parameter; ξ_i, η components of operators transforming x^i, t, respectively. Then

$$x^{i\prime} = x^i + \tau\xi_i, \quad t' = t + \tau\eta$$

Since the conditions

$$ds'^2 = 0, \quad \frac{d^2x^{i\prime}}{dt'^2} = 0$$

have to be satisfied, by equating the coefficients of τ, we find from (2.1)

$$d\xi_j dx^j - c^2 d\eta dt = 0 \quad \text{since} \quad ds^2 = 0$$

or

$$\left(\frac{\partial\xi_\alpha}{\partial x^j}\, dx^j + \frac{\partial\xi_\alpha}{\partial t}\, dt\right) dx^\alpha - c^2 \left(\frac{\partial\eta}{\partial x^j}\, dx^j + \frac{\partial\eta}{\partial t}\, dt\right) dt - \lambda \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2 dt^2\right] = 0$$

Dummy indices of summation are used throughout. λ is a function of x^i, and t. The left hand side of the last equation is a quadratic form in coordinate and time differentials. Equating the symmetric coefficients of this form to zero, we obtain the first part of

* The maximum group G_m allowed by the Maxwell equations is greater than the group G_{15}. However, the group G_{15} will be the maximum of all possible subgroups of the group G_m whose non-identical transformations of t and x^i depend only on these variables themselves.

equations defining ξ_i and η

$$\frac{\partial \xi_i}{\partial x^i} = \lambda, \quad \frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i} = 0 \quad (i \neq j); \qquad \frac{\partial \xi_i}{\partial t} - c^2 \frac{\partial \eta}{\partial x^i} = 0 \qquad (i = 1, 2, 3) \quad (2.2)$$

If we now put

$$\frac{dx^i}{dt} \equiv p^i, \qquad \frac{d^2x^i}{dt^2} = q^i \tag{2.3}$$

and let the components of the operators for these quantities be ζ_{p^i}, ζ_{q^i} respectively, then

$$\zeta_{p^i} = \frac{d\xi_i}{dt} - p^i \frac{d\eta}{dt}, \qquad \zeta_{q^i} = \frac{d\zeta_{p^i}}{dt} - q^i \frac{d\eta}{dt} \tag{2.4}$$

Here, total differentiation with respect to t is carried out by virtue of (2.3). The invariance condition of the second set of equations from (2.1) will be satisfied if $\zeta_{q^i} = 0$, since

$$q^i = 0, \quad \sum_{i=1}^{3} (p^i)^2 - c^2 = 0$$

which on expansion becomes

$$\frac{\partial^2 \xi_i}{\partial t^2} - p^i \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^\alpha \partial x^\beta} p^\alpha p^\beta p^i + \left(\frac{\partial^2 \xi_i}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 \eta}{\partial t \, \partial x^\alpha} \delta_\beta{}^i\right) p^\alpha p^\beta +$$

$$+ \frac{\partial^2 \xi_i}{\partial t \, \partial x^\alpha} p^\alpha - \left(\mu^i_\gamma p^\gamma + \gamma^i\right) \left[\sum_{i=1}^{3} (p^i)^2 - c^2\right] = 0$$

($\delta_\beta{}^i$ is the Kronecker tensor; $\mu_\gamma{}^i$, and γ^i are some unknown functions of x^i and t). These relationships should be satisfied identically for all values of p. Equating the symmetric coefficients of this form, which is cubic in p, to zero, we find the second part of the equations defining ξ_i and η. By using these equations, it is easy to show that ξ_i and η may only be 2nd degree polynomials in x^i and t and should consequently be easily determined.

Without going into the details of the integration of the defining equations, we shall show that the most general solution of these equations contains 15 arbitrary linear constants*, and the same number of infinitesimal operators

$$X_1 = \frac{\partial}{\partial x^1}, \qquad X_2 = \frac{\partial}{\partial x^2}, \qquad X_3 = \frac{\partial}{\partial x^3}, \qquad X_4 = \frac{\partial}{\partial t}$$

$$X_5 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \quad X_6 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad X_7 = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2}$$

$$X_8 = c^2 t \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial t}, \quad X_9 = c^2 t \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial t}, \quad X_{10} = c^2 t \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial t}$$

$$X_{11} = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + t \frac{\partial}{\partial t}$$

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* Putting $\lambda = 1$, results in a complete Lorentz group.

$$X_{12} = [(x^1)^2 - (x^2)^2 - (x^3)^2 + c^2t^2] \frac{\partial}{\partial x^1} + 2x^1x^2 \frac{\partial}{\partial x^2} + 2x^1x^3 \frac{\partial}{\partial x^3} + 2x^1t \frac{\partial}{\partial t}$$

$$X_{13} = 2x^1x^2 \frac{\partial}{\partial x^1} + [-(x^1)^2 + (x^2)^2 - (x^3)^2 + c^2t^2] \frac{\partial}{\partial x^2} + 2x^2x^3 \frac{\partial}{\partial x^3} + 2x^2t \frac{\partial}{\partial t}$$

$$X_{14} = 2x^1x^3 \frac{\partial}{\partial x^1} + 2x^2x^3 \frac{\partial}{\partial x^2} + [-(x^1)^2 - (x^2)^2 + (x^3)^2 + c^2t^2] \frac{\partial}{\partial x^3} + 2x^3t \frac{\partial}{\partial t}$$

$$X_{15} = 2x^1t \frac{\partial}{\partial x^1} + 2x^2t \frac{\partial}{\partial x^2} + 2x^3t \frac{\partial}{\partial x^3} + [(x^1)^2 + (x^2)^2 + (x^3)^2 + c^2t^2] \frac{\partial}{\partial t}$$

The operators X_1 to X_7 (excluding X_4) correspond to the usual group of motions. The operators X_8 to X_{10} correspond to proper Lorentz transformations. The operator X_{11} yields the similarity transformation. The nonlinear operators X_{12} to X_{15} correspond to Moebius transformations [7]. To determine the final equations of the group G_{15} we ought to integrate the appropriate Lie equations. However, let us use a ready-made result borrowed from [7] (p. 482) in which the final equations of the group were obtained directly

$$x^{i'} = a_i + e^{\tau} \sum e_k a_{ik} \frac{x^k - \alpha_k \sum e_j (x^j)^2}{1 - 2\sum e_j \alpha_j x^j + \left(\sum e_j \alpha_j^2\right)\left(\sum e_l (x^l)^2\right)} \qquad (k, l, j, i = 0, 1, 2, 3) \tag{2.5}$$

$$e_0 = 1, \quad e_1 = e_2 = e_3 = -1, \quad x^\circ = ct, \quad a_{00}^2 = \frac{c^2}{c^2 - v^2}$$

The parameters a_{ij} are connected by the orthogonality conditions

$$\sum_{i=0}^{3} e_i a_{ik} a_{il} = e_k \delta_l{}^k, \qquad \sum_{i=0}^{3} e_i a_{ki} a_{li} = e_k \delta_l{}^k$$

which lead to

$$\frac{a_{0i}}{a_{00}} = \frac{v^i}{c}, \qquad \frac{a_{i0}}{a_{00}} = \frac{v^{i'}}{c}, \qquad a_{ik} = -\alpha_{ik} + (a_{00} - 1) \frac{v^i v^{k'}}{v^2} \tag{2.6}$$

which shall be used later.

Here the α_{ik} are the coefficients of the orthogonal transformation of the spatial coordinates and they retain their significance in the case under consideration; v^i and $v^{i'}$ are components of the invariant velocity and its inversion for the inertial system. Here they are assumed to be initial velocities.

3. Finding the conformally-invariant motions. The problem is posed as follows: it is necessary to find a family of motions possessing, generally speaking, velocities different from the velocity of light, and which would transform into itself under all transformations of the group G_{15} and only this group.

Let

$$x^{j'} = f^j(x, t, \alpha), \qquad t' = f(x, t, \alpha) \tag{3.1}$$

be finite transformations of the group G_{15}. We shall show that only the group G_{15} is allowed by the family of motions

$$f^j(x, t, \alpha) = a^j = \text{const} \tag{3.2}$$

Let the final equations of the group $G \supseteq G_{15}$ allowed by the family (3.2) have the form

$$x^{j'} = \Phi^j(x, t, \alpha'), \qquad t' = \Phi(x, t, \alpha') \tag{3.3}$$

As (3.2) are absolute invariants of the group (3.3), we should have

$$f^j\,[\Phi^i(x, t, \alpha'),\ \Phi(x, t, \alpha'), \alpha] = f^j\,[x, t, \varphi(\alpha', \alpha)]$$

Putting $\alpha = 0$, we find

$$\Phi^j(x, t, \alpha') = f^j\,[x, t, \varphi(\alpha', 0)]$$

Differentiating with respect to α' and putting $\alpha' = 0$, we find

$$\frac{\partial \Phi^j}{\partial \alpha'} = \frac{\partial f^j}{\partial \varphi^\varepsilon}\frac{\partial \varphi^\varepsilon(\alpha', 0)}{\partial \alpha'}, \qquad \xi^{*j}(x, t) = c_\varepsilon{}^j \xi^\varepsilon(x, t)$$

Here ξ^{*j} are components of the infinitesimal operators of the group G, and ξ^ε are components of the operators of the group G_{15}. Operators of the group G which do not enter into the group G_{15} (supplementary operators), are found from suitable linear combinations of the operators.

$$X^* = \eta^*(x, t)\frac{\partial}{\partial t} + \zeta^{*\gamma}(\alpha)\frac{\partial}{\partial \alpha^\gamma}$$

Let us now show that η^* are independent of x^j. Solving (3.3) for x^j (this is possible), we obtain

$$x^j = \varphi^j(t, \beta)$$

The quantities x^j are not transformed by the additional transformations, hence the equalities

$$X^*\varphi^j(t, \beta) = \eta^*(x, t)\frac{\partial \varphi^j(t, \beta)}{\partial t} + \zeta^\gamma(\beta)\frac{\partial \varphi^j(t, \beta)}{\partial \beta^\gamma} = 0$$

should be satisfied identically in all the variables. Hence

$$\eta^*(x, t) = -\zeta^\gamma(\beta)\frac{\partial \varphi^j(t, \beta)}{\partial \beta^\gamma} \Big/ \frac{\partial \varphi^j(t, \beta)}{\partial t}, \qquad \text{as} \qquad \frac{\partial \eta^*}{\partial x^i} = 0$$

It may now be shown that if $\eta^* \not\equiv$ const, then no additional transformations exist, which would, together with G_{15}, form a group.

In fact, let us consider, say, the operator $X_8 = c^2 t\partial/\partial x^1 + x^1\partial/\partial t$ belonging to the algebra corresponding to G_{15}, together with $X^* = \eta^*(t)\partial/\partial t$. Since $X_1 = \partial/\partial x^1$, $X_4 = \partial/\partial t$ are the only operators in this algebra containing only the variables x^1 and t, apart from X_8 itself, it follows that

$$(X_8, X^*) = x^1\eta^{*\prime}(t)\frac{\partial}{\partial t} - c^2\eta^*(t)\frac{\partial}{\partial x^1} = \alpha X_8 + \beta X_1 + \gamma X_4 + \eta_1^*(t)\frac{\partial}{\partial t} =$$
$$= \alpha\left(c^2 t\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial t}\right) + \beta\frac{\partial}{\partial x^1} + (\gamma + \eta_1^*(t))\frac{\partial}{\partial t}$$

Hence

$$x^1\eta^{*\prime}(t) = \alpha x^1 + \gamma + \eta_1^*(t), \quad -c^2\eta^*(t) = \alpha c^2 t + \beta \qquad \left(\eta^* = -\frac{\beta}{c^2}\right)$$

Thus we have proved that $G = G_{15}$. Let us now find the required law. Let us keep x^j fixed instead of $x^{j'}$ in (2.5). Furthermore, to simplify the calculations, let us put $x^1 = x^2 = x^3 = 0$ (this latter simplification does not, however lead to any loss of generality since the constants a', as is easily seen from the form of the transformations (2.5), are not essential).

Returning to the variable $t = x^\circ/c$, we obtain

$$x^{j'} = x_0^{i'} + e^\tau \frac{a_{i0}ct - c^2\gamma_i t^2}{1 - 2\alpha_0 ct + \beta c^2 t^2}, \qquad t' = t_0' + e^\tau \frac{a_{00}t - c\gamma_0 t^2}{1 - 2\alpha_0 ct + \beta c^2 t^2}$$
$$\left(\gamma_i = \sum_{k=0}^{3} e_k a_{ik}\alpha_k,\ \beta = \alpha_0^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2\right) \tag{3.4}$$

Our next problem is to eliminate t from the system (3.4), and to express the parameters appearing in the right-hand sides of equations, in terms of the initial conditions. Solving the last equation of (3.4) for t and substituting the result in the remaining equations of the system (3.4), we obtain after manipulation

$$x_0^{i'} = x_0^{i'} + \left(\frac{a_{i0}}{a_{00}}c - \frac{u_i\chi_1}{\chi_1^2 - a_{00}^2\chi_0}\right)(t' - t_0') +$$
$$+ \frac{a_{00}}{2}\frac{u_i}{\chi_1^2 - a_{00}^2\chi_0}\left(e^\tau a_{00} - \sqrt{4c^2\chi_0(t' - t_0')^2 + 4e^\tau c\chi_1(t' - t_0') + e^{2\tau}a_{00}^2}\right) \tag{3.5}$$
$$u_i = \gamma_i - \gamma_0\frac{a_{i0}}{a_{00}}, \qquad \chi_0 = \alpha_0^2 - \beta, \qquad \chi_1 = \alpha_0 a_{00} - \gamma_0 \qquad (i = 1, 2, 3)$$
$$\frac{dx^{i'}}{dt'} = v^i, \qquad \frac{d^2x^{i'}}{dt'^2} = w^i \qquad \text{as} \quad t' = t_0'$$

Analysis shows that the number of essential parameters in (3.5) is 9 (not considering t_0'). Omitting the elementary, but tedious, computations performed by utilizing (2.6), we arrive at the final result

$$u_i = -\frac{e^\tau a_{00}^2}{2c^2}w^i, \quad \chi_0 = \frac{e^{2\tau}a_{00}^4}{4c^4}\left(w^2 + \frac{a_{00}^2}{c^2}\varepsilon^2\right), \quad \chi_1 = \frac{e^\tau a_{00}^4}{2c^3}\varepsilon$$
$$(\varepsilon = w_1v_1 + w_2v_2 + w_3v_3)$$

Here ϵ is the scalar product of the vectors **v**, **w** (the initial velocity and acceleration). Substituting the found expressions for u_i, χ_0, and χ_1 into (3.5), we obtain the required law

$$x^{i'} = x_0^{i'} + \left(v^i - \frac{w^i\varepsilon}{w^2}\right)(t' - t_0') +$$
$$+ \frac{c^2w^i}{a_{00}^2w^2}\left[\left(\frac{a_{00}^2}{c^2}\left(w^2 + \frac{a_{00}^2}{2}\varepsilon^2\right)(t' - t_0')^2 + \frac{2a_{00}^2}{c^2}\varepsilon(t' - t_0') + 1\right)^{1/2} - 1\right] \tag{3.6}$$

If all the constants are eliminated from (3.6) by successive differentiation, we obtain the differential equations of motion of a material point

$$\frac{d^3x^{i'}}{dt'^3}+\frac{3}{c^2}\left(\sum_{j=1}^{3}\frac{dx^{j'}}{dt'}\frac{d^2x^{j'}}{dt'^2}\right)\left[1-\frac{1}{c^2}\sum_{j=1}^{3}\left(\frac{dx^{j'}}{dt'}\right)^2\right]^{-1}\frac{d^2x^{i'}}{dt'^2}=0 \quad (i=1,2,3) \quad (3.7)$$

Under the condition

$$\sum_{j=1}^{3}\left(\frac{dx^{j'}}{dt'}\right)^2 \neq c^2$$

the set of equations (3.7) determines the respective invariant of the group G_{15} continued up the third derivative (of the coordinates with respect to time) inclusive. This can be checked directly.

A more direct way of determining the law (3.6) is the following: Having suitably continued the group G_{15} it is possible to find its respective invariant under the condition

$$\sum_{j=1}^{3}\left(\frac{dx^{j'}}{dt'}\right)^2 - c^2 \neq 0$$

and then to integrate differential equations obtained. In practice this method is found to be laborious.

4. Kinematic properties of conformally-invariant motions of a material point.

(1) Since the radicand in (3.6) remains positive, the law of motion has meaning only if the discriminant of this radicand is negative $-a_{00}^2 w^2/c^2 \leqslant 0$. The latter is true if and only if $a_{00}^2 \geqslant 0$, i.e. $v^2 \leqslant c^2$. This mathematical fact may be interpreted as an indication of the impossibility of motions possessing velocities greater than that of light.

(2) If the projection of the initial acceleration on any of the coordinate axes is equal to zero $w^1=0$, say, then $d^2x^{1'}/dt'^2=0$ at any instant during the motion. If all three components of the initial acceleration are zero, the motion of the material point always remains rectilinear and uniform.

(3) Let the initial acceleration be different from zero $w \neq 0$. Differentiating (3.6) we obtain in the limit

$$\lim_{t'\to\infty}\frac{dx^{i'}}{dt'}=v^i-\frac{w^i\varepsilon}{w^2}+\frac{w^i c}{a_{00}^2 w^2}\left(w^2+\frac{a_{00}^2}{c^2}\varepsilon^2\right)^{1/2}=\text{const}, \quad \lim_{t'\to\infty}\sum_{i=1}^{3}\left(\frac{dx^{i'}}{dt'}\right)^2=c^2$$

i.e. if the initial acceleration of the point is not equal to zero, then its velocity approaches asymptotically the limiting velocity.

Let us differentiate (3.6) twice and let us form the expression for the square of the modulus of the acceleration

$$\sum_{i=1}^{3}\left(\frac{d^2x^{i'}}{dt'^2}\right)^2=w^2\left[\frac{a_{00}^2}{c^2}\left(w^2+\frac{a_{00}^2}{c^2}\varepsilon^2\right)(t'-t_0')^2+2\frac{a_{00}^2}{c^2}\varepsilon\,(t'-t_0')+1\right]^{-3}$$

We can see from this expression that for $w \neq 0$ the modulus of the acceleration begins to decrease and tends to zero, at some time $t' < t_0'$, so that

$$\max \sum \left(\frac{d^2 x^{i'}}{dt'^2}\right)^2 = w^2 \quad (t' \geqslant t_0')$$

(4) The trajectory of the motion is a hyperbola in the plane formed by the initial velocity and acceleration vectors. When these vectors are parallel, the hyperbola becomes a straight line.

(5) The universe of the Maxwell equations described by the group G_{15} does not contain any universal (therefore untransformable) constants except c. Hence, in conformity with the viewpoint developed here, it is necessary to separate out of the family of motions, for which the group G_{15} is the group of all their automorphisms, such a subgroup, which would not contain untransformable constants different from c. Such a family is given by (3.6). It is easy to show that in the above mentioned sense it is the most general one.

Actually, the law (3.7), being a differential expression of the law (3.6), is generated by the complete set of invariants of the triply continued group G_{15}. This means that this group has no other invariants. Hence, if a law existed which was more general than (3.7), it should be looked for amongst the invariants of the group G_{15} continued to at least four derivatives. But such a group operates in a space of 16 dimensions, which exceeds the order of the group, hence, it is intransitive and, therefore, allows absolute invariants. The absolute invariants contain untransformable constants which are essential to the solution. This however contradicts our previous statements. The law (3.6) is also an unique one since there are no two different systems of differential invariants of the same order, each of which would uniquely determine the same group.

5. On the treatment of conformally-invariant motions. The following alternative arise in the discussion of the obtained results.

(1) Conformally-invariant motions do not occur in nature for any initial values of the acceleration (except zero). In this case it is necessary either to assume that the description of space-time given by the Maxwell equations in terms of the physical concepts entering these equations is not complete; or to cast a doubt on the validity of the invariance principles.

(2) Conformally-invariant motions do occur in nature for values of the initial accelerations different from zero. In this case they may occur either for arbitrary initial conditions, i.e. the point may be considered isolated or, for some fixed (different from zero,) initial accelerations, i.e. in a force field. Let us discuss these possibilities.

(*a*) *Motion of an Isolated Point.* (This has already been partially discussed in section 1). The Galileo inertial law is satisfied in nature with a high degree of accuracy. The conformally-invariant motions (3.6) do not contradict it.

Indeed, the Galileo Inertial Law states that if at a particular instant a material point is at rest or in the state of uniform motion (instantaneous acceleration equal to zero), then it will remain in this state all the time. But this also follows from (3.6). Nothing is said in the Galileo inertia law about the pre-history of such instantaneous states, not any alternatives offered. Hence, the following deduction may be formulated.

The Galileo Inertia Law is valid in the universe of the Maxwell equations. However, if an instantaneous state in which the acceleration is not zero is possible for a free material point, then this acceleration will not vanish instantaneously, but it will tend asymptotically to zero. At the same time, a point will be accelerated to the velocity of light, and this will be independent of its initial velocity.

b) Motion in a Force Field. Only two kinds of mechanical force interactions are known: Newtonian and relativistic; after a single integration, (3.7) may be rewritten for them as

$$m_0 \frac{d\mathbf{V}}{dt} = \left(1 - \frac{V^2}{c^2}\right)\mathbf{h}, \qquad \frac{d\mathbf{p}}{dt} = \left(1 - \frac{V^2}{c^2}\right)\mathbf{h} + \frac{1}{c^2}(\mathbf{h}\cdot\mathbf{V})\,\mathbf{V} \qquad (\mathbf{h} = a_{00}{}^{3/2} m_0 \mathbf{w}) \quad (5.1)$$

Here m_0 is the rest mass of a material point, $\mathbf{h}$ is a constant vector; $\mathbf{p}$ is the relativistic momentum. Since the right-hand sides of (5.1) do not generally agree with the Lorentz force, conformally-invariant motions cannot be generated only by an electromagnetic field alone.

Analysis shows that motions described by (5.1) may occur only for particular solutions of the Maxwell equations (for example, $\mathbf{H} = 0$, $\mathbf{E} = \text{const.}$). At the same time, a particular motion belonging to (3.6) is obtained (initial velocity should be collinear with direction of the field.

These motions cannot be generated by an Einsteinian gravitational field. This is seen directly from the fact that the gravitational field should bend light rays, which is not the case. Hence, attempts to interpret the right hand sides of (5.1) as the component of any of the known fields apparently encounter serious difficulties, within the scope of the mentioned definitions of forces.

Consequently, the author is inclined to accept the first of the treatments, according to which the conformally-invariant motions may be performed by an isolated point and, are therefore inertial.

BIBLIOGRAPHY

1. Wigner, E.P., Symmetry and Conservation Laws. Phys. Today, 1964, Vol. 17, No. 34 (Russian translation: Wigner, E. Simmetriia i zakony sokhraneniia. Usp. fiz. nauk, 1965, Vol. 83, No. 4).

2. Wigner, E. Sobytiia, zakony prirody i printsipy invariantnosti (Events, laws of nature and invariance principles). Usp. fiz. nauk, 1965, Vol. 85, No. 4.

3. Einstein, A., Sushchnost' teorii otnositel'nosti (Substance of relativity Theory). Moscow, Izd. inostr. lit., 1955.

4. Pauli, W., Teoriia otnositel'nosti (Relativity). Moscow-Leningrad, Gostekhizdat, 1947.

5. Berestetskii, V.B., Dinamicheskie simmetrii sil'no vzaimodeistvuiyshchikh chastits

(Dynamical symmetries of strongly interacting particles). Usp. fiz. nauk, 1965, Vol. 85, No. 3.

6. Utiyama, R. Invariant theoretical interpretation of interaction. Phys. Rev. 1956, Vol. 101, No. 5, 1597 - 1607.

7. Fok, V.A., Teoriia prostranstva, vremeni i tiagoteniia (Theory of space, time and gravitation). Moscow, Gostekhizdat, 1955.

8. Liouville, J. Math., 1847, Vol. 12, Note VI, p. 265.

9. Lie, S., Scheffers, G., Geometrie der Berührungstransformationen. Bd. 1, Leipzig 1896.

10. Cunningham, E., Proc. London Math. Soc., 1910, Vol. 8, 77 ; Bateman, H., Proc. London Math. Soc., 1910, Vol. 8, 223.

11. Klein, F., Vorlesungen über die Entwicklung der Mathematik im 19 Jahrhundert. Bd.2, Berlin, Springer, 1927.

12. Page, L., A New Relativity. Phys. Rev., 1936, Vol. 49, No. 3, p. 254.

13. Milne, E.A., Relativity, Gravitation and World-Structure. Clarendon Press Oxford, 1935.

14. Caratheodory, C., Zur Axiomatik der Speziellen Relativitäts Theorie. Sitzungs Berichte der Preussichen Akademi der Wissenschaften Phys.- Math. Klasse, 1924, p. 12-27 (Russian translation; Karateodori, K., K aksiomatike spetsial'noi teorii otnositel'nosti. *Razvitie sovremennoi fiziki,* Moscow, Izv. 'Nauka', 1964, pp. 167-187).

15. Fulton, T, Rohrlich, R., Witten, L.: Conformal Invariance in Physics. Rev. Modern Physics., 1962, Vol. 34, No. 3, pp. 442-457.

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